Determination of a Missile Polar Mass Moment of Inertia

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Introduction

THE polar mass moment of inertia J_{xx} with respect to the longitudinal concentric centerline of a typical missile is important for flight dynamics and control. One of the extensively used methods for measuring J_{xx} is by suspending it on knife-edge type supports with thin steel wire ropes as shown in Figs. 1 and 2 and then observing its free oscillation period under gravity.[1,2] Subsequently, J_{xx} can be quickly calculated with the observed small-angle period T, associated physical properties, and dimensional data neglecting the minimal mass of the thin steel ropes

$$J_{xx} = \left(mgh\frac{T^2}{4\pi^2} - mh^2\right) \tag{1}$$

where m is the mass of the missile and h the distance between the center of the missile and the center of the supporting point (or the center of the supporting circle in a later discussion). Superficially, the aforementioned experimental method for computing the polar mass moment of inertia is seemingly perfect. Despite the rather high accuracy requirement by aeronautical industries, the real-life laboratory determination of J_{xx} by the preceding method generally and persistently gives numerical data with sizable errors when compared with known J_{xx} data of simple steel cylinders. In this study, the major culprit for the inaccuracy of test results is found to be the difficulties in modeling the so-called knife-edge supports. In real-life laboratory tests, to avoid extremely high stress at the contact points between the wire ropes and knife-edge supports, the sharp edge of the support is generally replaced by some rounded contact portion with small diameters of 1 or 2 cm as shown in Fig. 3. Although this slight modification of the sharp-edge support is seemingly negligible in the whole testing procedure, nevertheless studies show that this minor and practical modification of the support situation does introduce sizable errors in evaluating the polar mass moment of inertia of missiles. Surprisingly, for typical ship-to-ship missiles and air-to-air missiles, the errors may range up to 15% or even 30% below the correct value, primarily depending on diameters of the testing missile and the supporting circle.

Dynamics Analysis Accommodating the Support Condition

With the conservation of energy concept,[3,4] a dynamics analysis that accommodates the detailed support is developed for upgrading the accuracy of J_{xx}. In Fig. 4, the lower large cylinder with radius R_1 represents the front view of a testing missile, which is suspended by thin steel wire ropes on a fixed circular rod as represented by the upper small circle R_2. When the R_1 cylinder oscillates freely along the center of the small R_2 supporting circle, the motion of the R_1 cylinder includes combined translation and rotation behavior. Since the thin wrapping steel ropes remain slipless on the upper surface of the supporting circle during the oscillating process,

$$R_2\theta_1 = R_1\phi \tag{2}$$

where angular displacements θ_1 and ϕ are shown in Fig. 4. Then

$$\phi = \frac{R_2}{R_1}\theta_1 \tag{3}$$

Consequently, θ_1 and θ_2 as shown in Fig. 4 are related by

$$\theta_2 = \theta_1 - \phi = \theta_1 - \frac{R_2}{R_1}\theta_1 = \left(1 - \frac{R_2}{R_1}\right)\theta_1 \tag{4}$$

Note that θ_2 represents the true angular displacement of the test missile. For small-angle oscillations neglecting the mass of the rope, the total kinetic energy E_k of the missile can be expressed as

$$E_k = 1/2m(h\dot{\theta}_1)^2 + 1/2J_{xx}\dot{\theta}_2^2 \tag{5}$$

or in terms of θ_1

$$E_k = 1/2mh^2\dot{\theta}_1^2 + 1/2J_{xx}\theta\left(1 - \frac{R_2}{R_1}\right)^2\dot{\theta}_1^2 \tag{6}$$

On the other hand, the potential energy U of the missile can be written as

$$U = mgh(1 - \cos\theta_1) \tag{7}$$

or in terms of power series

$$U = mgh\left(\frac{\theta_1^2}{2} - \frac{\theta_1^4}{4!} + \dots\right) \tag{8}$$

For small-angle oscillations, Eq. (8) can be closely approximated as

$$U = mgh\frac{\theta_1^2}{2} \tag{9}$$

Assuming a conservative system, then

$$\frac{dE_k}{dt} + \frac{dU}{dt} = 0 \tag{10}$$

or

$$1/2mh^2 2\dot{\theta}_1\ddot{\theta}_1 + 1/2J_{xx}\left(1 - \frac{R_2}{R_1}\right)^2 2\dot{\theta}_1\ddot{\theta}_1 + mgh\dot{\theta}_1\theta_1 = 0 \tag{11}$$

or

$$\left[mh^2 + \left(1 - \frac{R_2}{R_1}\right)^2 J_{xx}\right]\ddot{\theta}_1 + mgh\theta_1 = 0 \tag{12}$$

From the solution to this equation, the small-angle natural frequency of the R_1 cylinder can be obtained as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{mgh}{[1 - (R_2/R_1)]^2 J_{xx} + mh^2}} \tag{13}$$

where m is the mass of the R_1 cylinder and h the center-to-center distance between large and small circles. Conversely, the mass moment of inertia of the R_1 circle can be easily obtained as

$$J_{xx} = \frac{1}{[1 - (R_2/R_1)]^2}\left(mgh\frac{T^2}{4\pi^2} - mh^2\right) \tag{14}$$

where T is the free oscillation period. For a true knife-edge support, the denominator of Eq. (14) is unity, since R_2 approaches zero. But in the practical laboratory environment, say $R_2 = 1.0$ cm, and $R_1 = 15.0$ cm for a typical ship-to-ship missile, the traditional

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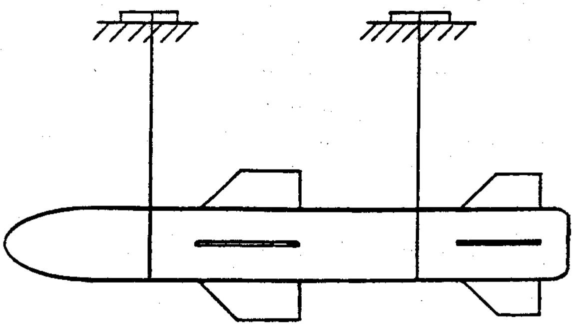

Fig. 1 **J_{xx} testing with knife-edge support (side view).**

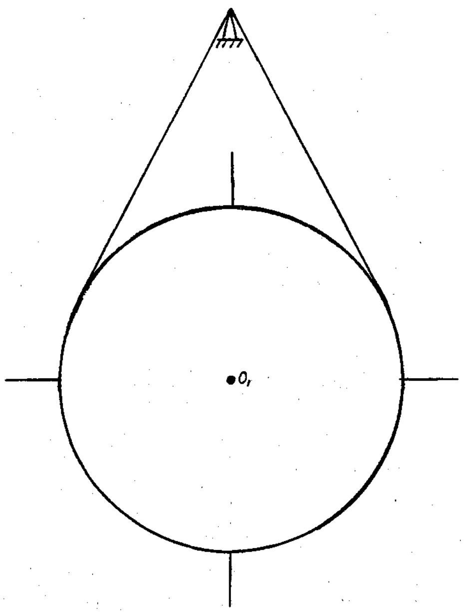

Fig. 2 **J_{xx} testing with knife-edge support (front view).**

Eq. (1) will give a J_{xx} about 15% below the true value as demonstrated in Eq. (14). Moreover, for a typical air-to-air missile generally with smaller radius, say 8 cm, the traditional equation for calculating the polar J could be 30% below the correct value.

It is interesting to note that the system shown in Fig. 4 will demonstrate a minimum period, whereas the center-to-center distance h is as follows:

$$h = k_0^* = \sqrt{\frac{[1-(R_2/R_1)]^2 J_{xx}}{m}} = \left(1-\frac{R_2}{R_1}\right)k_0 \qquad (15)$$

where k_0^* is a modified value of the radius of gyration k_0 of the R_1 circle in testing. Equation (15) was derived by considering the equation of period and differentiating with respect to h. This h can be recognized to be the center-to-center distance for the minimum period of the system. As the practical experimental situation allows, the distance h should be kept as short as possible to reduce the error involved in obtaining J_{xx} due to inaccuracy in the temporal reading of periodic duration. Experiences show that an improper selection of h will easily give a J_{xx} with large error. On the other hand, except with a special arrangement of clamping and oscillating apparatus, the radius of the test missile will limit the minimum value of h.

Experimental Example

Experimental verification can be demonstrated by testing a uniform thin-walled steel cylinder simulating the casing of a Gabriel-class ship-to-ship missile booster. The physical data of such a uniform cylinder made of SAE 4140 steel are

Outer radius:

$$R_1 = 0.173 \text{ m}$$

Inner radius:

$$R_i = 0.1589 \text{ m}$$

Length: (16)

$$L = 0.588 \text{ m}$$

Mass of cylinder:

$$m = 67.6 \text{ kg}$$

Local gravity:

$$g = 9.7888 \text{ m/s}^2$$

The J_{xx} of such a homogeneous thin cylinder can be conveniently calculated with high accuracy by

$$J_{xx} = \tfrac{1}{2} m \left(R_1^2 + R_i^2\right) = 1.8647 \text{ kg-m}^2 \qquad (17)$$

In this test, the radius of the circular supporting rod is $R_2 = 0.0131$ m, and the center-to-center distance h between the R_1 and R_2 circles

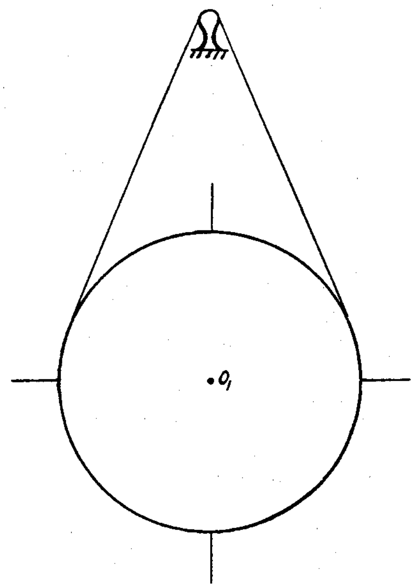

Fig. 3 **Rounded knife-edge support.**

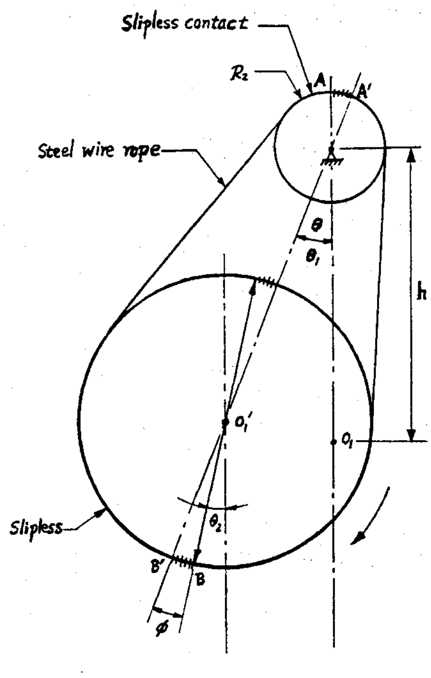

Fig. 4 **J_{xx} testing with fixed rod support.**

is appropriately arranged as 0.2675 m. For a maximum angle of oscillation θ_0 below 40 deg, the nonlinear effect on the oscillation period is minimal (less than 0.1% difference from the exact value)[5]; thus Eq. (14) can be appropriately applied to obtain the J_{xx}. For this test, the average period of 50 cycles $T = 1.197$ s is used in Eq. (14) to obtain J_{xx} as 1.881 kg-m^2, which is less than 1% difference from the accurately calculated value of 1.8647 kg-m^2. On the other hand, if Eq. (1) instead of Eq. (14) is employed for this calculation, it can be easily seen that the error involved would be around 15% below the true J_{xx} value of the tested cylinder. Thus Eq. (14) shows a significant improvement in accuracy over the traditional Eq. (1).

Conclusion

Although the minor modification of the support configuration for an oscillating test seems negligible, it can actually degrade the accuracy of J_{xx} seriously. Since the knife-edge support for an oscillation test cannot be fully realized, it would be necessary to reanalyze the theoretical side to accommodate the practical experimental difficulties, otherwise noticeable errors will occur. Furthermore, for obtaining even higher accuracy of J_{xx} at relatively large test oscillation angles, it would be feasible to take into consideration the nonlinear effect on the oscillation period.[5] Incidentally, the condition for the minimum period of such a special compound pendulum has been derived. It is closely related to the radius of gyration of the test missile; in conjunction with other test parameters, these data are useful in adjusting the center-to-center distance h to optimize the test conditions.

On the other hand, with some advantages and disadvantages, other test methods may also be applied to obtain J_{xx}. For example, a missile may be assembled as a compound pendulum with appropriate fixture and supporting roller bearings; then the simple torsional dynamics equation can be used to obtain J_{xx} of the whole oscillating system and subsequently the J_{xx} of the test missile.

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New Design Formulas for a Flex-Bearing Joint

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Introduction

FLEX bearing is a nonrigid, pressure-tight connection between the rocket motor and movable nozzle that allows the movable nozzle to deflect in a specified direction. Solid propellant rocket stages are generally equipped with these flexible joints to provide thrust vector control. These joints usually consist of several conical/spherical rings of elastomeric material alternating with conical/spherical reinforcements as shown in Fig. 1.

Akiba et al.[1] presented the development and test results of spherical flex bearings. Design, analysis, and testing details of flex bearings for various configurations of reinforcements and elastomers are available in Refs. 2–5. The design formulas for the compressive hoop stresses in the reinforcements, proposed in Refs. 2 and 5 are derived from the test results of joints varying from 19.3 to 56 cm in diameter and hence are valid only in this range. Based on these formulas, a flex-bearing joint was designed. Subsequently, the geometrically nonlinear finite element analysis was carried out using the general-purpose program MARC. The analysis results were found to be much different compared with the design formulas assessment.[2–5] Later, the test results confirmed the analysis predictions. Furthermore, the design formulas[2,5] for the compressive hoop stresses are seen to be highly sensitive to reinforcement thickness, and the stresses so obtained are linearly proportional to the motor pressure, whereas the analysis and test results did not show this trend. These inadequacies in the existing formulas motivated the authors to develop refined formulas to have accurate predictions of the compressive hoop stresses at the inner radii of the reinforcements. The refined formulas are developed based on the analyses and test results of the following configurations of flex-bearing joints: 1) six elastomer pads (of natural rubber formulation), each of 3.0 mm thickness, and six 15-cdv-6 steel reinforcements, each of 3.0 mm thickness; and 2) six elastomer pads, each of 3.0 mm thickness, and the thickness of 15-cdv-6 steel reinforcements (in millimeters) is varied as 4.5 /5.5 /5.5 /5.5 /5.5 /4.5.

The effectiveness of these formulas are shown through test results. Since the proposed formulas are an improvement over the existing formulas (Refs. 2 and 5), the applicability and validity range of the formulas remain the same, that is, joints having diameters within 19.3–56 cm.

Existing Design Formulas

The stresses in the reinforcements are tensile hoop on the outer radius and compressive hoop stress on the inner radius due to motor pressure and actuation loads.[2,5] In Ref. 5 the following formulas are suggested to compute the compressive hoop stress due to motor pressure and actuation loads:

$$\sigma_P = \frac{4087}{n-1} P_C K_R \Omega \tag{1}$$

$$\sigma_V = \frac{43{,}950}{n-1} \theta K_R \Omega \tag{2}$$

with

$$\Omega = \frac{R_P^{2.4}\cos\beta}{3283 t_R^3 + t_R \cos^2\beta[R_P^2(\beta_2 - \beta_1)^2 - 3283 t_R^2]}$$

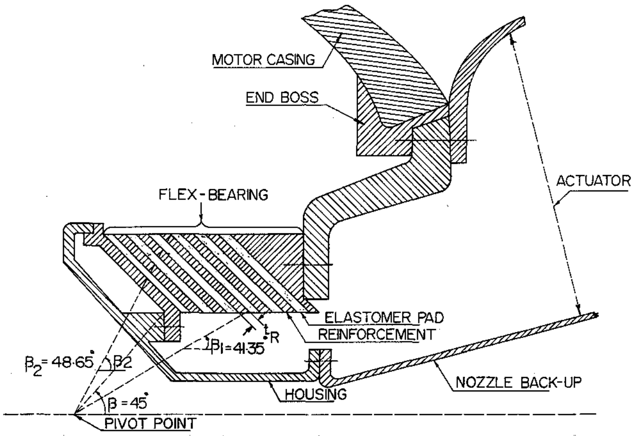

Fig. 1 Typical flex bearing in position with other subassemblies.

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